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Some Real Interpolation Methods for Families of Banach Spaces: A Comparison

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We consider certain real interpolation methods for families of Banach spaces. We also define some new such methods obtained by passing to the limit in the constructions of Sparr and Cobos–Peetre. The relations between all these methods are studied. Characterizations of minimal and maximal spaces are obtained. Some concrete examples as well as sharp estimates of the corresponding operator norms are also exhibited. © 1997 Academic Press

0. INTRODUCTION

In the theory of interpolation one usually considers Banach couples, i.e., pairs (A_0, A_1) such that A_0 and A_1 are Banach spaces embedded in a

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common topological vector space $\mathcal{U}$. The most important among the various constructions of interpolation spaces with respect to a given couple is the complex method leading to spaces $[A_0, A_1]_\theta$ (where $0<\theta<1$), and the real method leading to spaces $(A_0, A_1)_{\theta,q}$ (where $0<\theta<1$, $0<q\leqslant\infty$). See e.g. the books [1, 2, 17] and the bibliography by Maligranda [19] (including approximately 2500 references).

Part of the theory of interpolation between two Banach spaces can be generalized to cover situations where one interpolates between finitely many Banach spaces and even between general families of (infinitely many) Banach spaces. In this direction let us mention the following developments:

(1) A theory of *complex interpolation* between families of Banach spaces was developed by Coifman *et al.* (see [8–10]) and, independently, by Kreĭn and Nikolova (see [15, 16]). These spaces are often referred to as the St. Louis and the Voronezh spaces respectively. Another complex interpolation method between n-tuples of Banach spaces was suggested by Lions [18] and studied in detail by Favini [12]. The Favini–Lions theory was extended by Cwikel and Janson [11] to cover also complex interpolation between very general families of spaces.

(2) A theory of *real interpolation* between n-tuples of Banach spaces was worked out by Sparr [24]. A parallel theory of interpolation between 2^n-tuples of Banach spaces was studied by Fernández [13]. In this connection we mention also early work by Foiaş and Lions, Kerzman, and Yoshikawa (cf. the discussion in [24, p. 248]). Lately Cobos and Peetre [7] have developed a theory which, in particular, covers both the constructions of Sparr and Fernández with $n=3$, respectively $n=4$. On the other hand, even earlier the construction of Sparr had been extended by Cwikel and Janson [11] to the case of interpolation between a fairly general family $A=\{A_t\}_{t\in\Gamma}$, where the A_t are Banach spaces and Γ is a general probability space.

In this paper we consider, in particular, certain real interpolation methods recently studied in [3] and [22] (see also [4, 20, and 23]). We introduce also new methods obtained by applying a limiting process in the constructions in Sparr [24] and Cobos–Peetre [7]. In all these cases we are in the situation when the actual family of Banach spaces is indexed by the points of the unit circle $\mathbb{T}=\{|z|=1\}$ in the complex plane $\mathbb{C}$, while the interpolation spaces are labelled by points of the unit disk $D=\{|z|<1\}$. Relations between (K- and J-versions of) these methods and the method of Cwikel and Janson are discussed. Characterizations of minimal and maximal spaces are obtained (these results are applicable for families of complex as well as of real spaces). We also include some exact computations of a certain function $D_{z_0}(M)$, which yields sharp estimates of the corresponding operator norm (and informally referred to as the "Dicesar function").

The paper is organized in the following way. All definitions of the families of interpolation spaces studied and other necessary preliminaries are collected in Section 1. In Section 2 we prove some general results including a sharp embedding result and also the characterization of the natural maximal and minimal interpolation spaces. In Section 3 comparison results are established. In Section 4 we present some calculations of the function $D^S_{z_0}(M)$ for the sets S considered in this paper. Some further examples are given in Section 5.

Convention. If X is a Banach space, its norm will be written $\|\cdot\|_X$ or sometimes $\|\cdot; X\|$ (if the symbol for the space is very complicated).

1. PRELIMINARIES

As in the Introduction, let D and $\mathbb{T}$ be the unit disc respectively the unit circle. We say that the triple $\bar{A} = \{A(\gamma)\colon \gamma \in \mathbb{T}; \mathscr{A}; \mathscr{U}\}$ is an *interpolation family* on $\mathbb{T}$ with $\mathscr{U}$ as the *containing space* (assumed to be a Banach space) and with $\mathscr{A}$ as the *log-intersection space* (in the sense of Coifman–Cwikel–Rochberg–Sagher–Weiss) if the following conditions hold:

(a) each $A(\gamma)$ is a Banach space continuously imbedded in $\mathscr{U}$ (we shall denote the norm in $A(\gamma)$ by $\|\cdot\|_\gamma$ and the one in $\mathscr{U}$ by $\|\cdot\|_{\mathscr{U}}$);

(b) for each $a \in \bigcap_{\gamma \in \Gamma} A(\gamma)$ the assignment $\gamma \mapsto \|a\|_\gamma$ gives a measurable function on $\mathbb{T}$;

(c) $\mathscr{A}$ coincides with the set of elements $a \in \mathscr{U}$ such that $a \in A(\gamma)$ a.e. on $\mathbb{T}$ with $\int_{\mathbb{T}} \log^+ \|a\|_\gamma \, d\gamma < \infty$; moreover, it is assumed that there exists a measurable function P on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \log^+ P(\gamma)\, d\gamma < \infty \text{ and } \|a\|_{\mathscr{U}} \leqslant P(\gamma)\|a\|_\gamma \qquad \text{a.e. on } \mathbb{T} \text{ for } a \in \mathscr{A}.$$

If $\bar{A}$ is a *bounded family*, i.e. $P(\gamma) = 1$ for all $\gamma \in \mathbb{T}$, then the following *K-functional* was defined [21] (see also [20, 23]):

$$K_1(\alpha, a) = \inf\left\{\sum_j \alpha(\gamma_j)\|a_j\|_{A(\gamma_j)}\right\},$$

where $\alpha\colon \mathbb{T} \to \mathbb{R}^+$ is a given measurable function and the infimum is taken over all representations of the element a as an infinite sum $a = \sum_j a_j$ in $\mathscr{U}$ with $a_j \in A(\gamma_j)$, $\gamma_j \in \mathbb{T}$.

The *sum space* $\sum_\gamma A_\gamma$ of a bounded family of Banach spaces $\{A_\gamma\}$ is defined as the set of all elements $a \in \mathscr{U}$ that can be written as $a = \sum_\gamma a_\gamma$

with $a_\gamma \in A_\gamma$ and $\sum_\gamma \|a_\gamma\|_{A_\gamma} < \infty$. Clearly $\sum_\gamma A_\gamma$ is a Banach space with the norm $a \mapsto \|a\| = \inf \sum_\gamma \|a_\gamma\|_{A_\gamma}$, where the infimum is taken over all representations of a of the form $a = \sum_\gamma a_\gamma$, $a_\gamma \in A_\gamma$ (see [17]); note that only countably many summands a_γ are different from zero.

Another K-functional was defined in [3]:

$$K_2(\alpha, a) = \inf \left\{ \int_{\mathbb{T}} \alpha(\gamma) \|a(\gamma)\|_\gamma \, d\gamma \right\},$$

where the infimum is taken over all representations of the element a as an integral $a = \int_\Gamma a(\gamma) \, d\gamma$ with $a(\cdot) \in \bar{G}$. Here G denotes the set of functions $b = b(\cdot)$ of the form $b = \sum_j b_j \chi_{Ej}$ with $b_j \in \mathscr{A}$, the E_j being pairwise disjoint measurable sets of $\mathbb{T}$, while $\bar{G}$ stands for the set of all Bochner integrable functions $a(\cdot)$ with values in $\mathscr{U}$ such that $a(\gamma) \in A(\gamma)$ a.e. on $\mathbb{T}$ which can be approximated pointwise a.e. in the $A(\cdot)$-norm by a sequence of functions $a_n(\cdot)$ belonging to G, that is, we have

$$\|a_n - a(\gamma)\|_{A(\gamma)} \to 0 \qquad \text{a.e.}$$

With no loss of generality one can assume that $\|a_n(\gamma)\|_{A(\gamma)} \leqslant C \, \|a(\gamma)\|_{A(\gamma)}$ with a constant $C > 1$.

We shall informally refer to these two cases as the "discrete" case ($j = 1$) and the "continuous" case ($j = 2$) respectively.

Next, let us fix a multiplicative subgroup S of (if $j = 1$) bounded or (if $j = 2$) essentially bounded functions. For $z_0 \in D$ and $1 \leqslant p \leqslant \infty$ we can then define the following interpolation spaces:

$$(A)^{S, j}_{z_0, p; K} = \left\{ a \in \mathscr{U} : \left(\sum_{\alpha \in S} \left(\frac{K_j(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} < \infty \right\},$$

where $\alpha(z_0) = \exp(\int_{\mathbb{T}} \log \alpha(\gamma) \, P_{z_0}(\gamma) \, d\gamma)$, while $P_{z_0}(\gamma)$ is the Poisson kernel. For $j = 2$ these spaces were investigated in [3, 4].

In the sequel, in order to simplify the notation, whenever expedient, we shall write $(A)^{S,j}_K$ in place of $(A)^{S,j}_{z_0, p; K}$. Moreover, we shall sometimes suppress the index j writing just $(A)^S_K$; thus in such cases the symbol $(A)^S_K$ refers to any of the spaces $(A)^{S,1}_K$ and $(A)^{S,2}_K$ (cf. [21, 3]).

The analogous *J-functional* is defined as follows

$$J(\alpha, a) = \operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \alpha(\gamma) \|a\|_\gamma$$

for $a \in A(\gamma)$ a.e. and $\alpha(\gamma) \in L^\infty(\mathbb{T})$.

Throughout this paper we shall make the auxiliary assumption that $J(\alpha, a) < \infty$ for every a in the intersection $\mathscr{A}$ and all $\alpha \in S$.

The corresponding interpolation spaces are defined as follows:

$$(A)^S_{z_0,p;J}=\Big\{a\in\mathcal{U}\colon a \text{ can be represented in the form } a=\sum_{\alpha\in S} a_\alpha,$$
$$\text{where } a_\alpha\in\mathcal{A} \text{ and the expression } \Big(\sum_{\alpha\in S}\Big(\frac{J(\alpha,a_\alpha)}{\alpha(z_0)}\Big)^p\Big)^{1/p}<\infty\Big\}.$$

The norm is defined as the infimum of all such expressions. Again, whenever possible we shall write just $(A)^S_J$ for $(A)^S_{z_0,p;J}$. These spaces were introduced in [3, 4]. A similar theory can be developed corresponding to $j=1$. In this case the J-functional is defined by

$$J_1(\alpha,a)=\sup_\gamma \alpha(\gamma)\|a\|_\gamma,$$

where α is a bounded function on $\mathbb{T}$.

Remark 1.1 (change of notation). We have thus changed the notation compared to [3]. There these K and J-spaces were (for $j=2$) written $[A]^S_{z_0,p}$ respectively $(A)^S_{z_0,p}$. We have given preference to the present notation, because in this way it is easy to remember which one of them is associated with the K-functional and which one with the J-functional.

Moreover, we let ΔA be the vector space $\mathcal{A}$ equipped with the norm $J(1,a)$ and put $K_j(\bar{A})=\{a\in\mathcal{U}\colon K_j(1,a)<\infty\}$. Clearly, we have $K_1(\bar{A})=\Sigma\bar{A}$. If we say that a statement holds for $K(\bar{A})$ we mean that it holds for both $K_1(\bar{A})$ and $K_2(\bar{A})$.

For any positive measurable function M on $\mathbb{T}$ we set

$$D^S_{z_0}(M)=\inf_{\alpha\in S}\Big\{\operatorname*{ess\,sup}_{\gamma\in\mathbb{T}}\frac{M(\gamma)\,\alpha(\gamma)}{\alpha(z_0)}\Big\}\qquad\text{(Dicesar function)}.$$

Here and later on "ess sup" shall be interpreted as "sup" in the discrete case. Moreover, in the sequel "ess inf" and "a.e." shall be interpreted as "inf" and "everywhere", respectively, when working with the $(\text{—})^1$ method.

We recall (see [4] and Section 4 of this paper) that for the norm of an interpolated operator $T\colon F(\bar{A})\to F(\bar{B})$ such that $\|T\|_{A(\gamma)\to B(\gamma)}\leqslant M(\gamma)$ a.e. on $\mathbb{T}$ we have the upper estimate

$$\|T\|_{F(\bar{A})\to F(\bar{B})}\leqslant D^S_{z_0}(M),$$

provided the functor $F=F(\cdot)$ is either $(\cdot)^S_K$ or $(\cdot)^S_J$.

2. GENERAL RESULTS

2.1. *Maximal and Minimal Interpolation Conditions*

DEFINITION 2.1. We say that a Banach space A belongs to the *class* $\mathfrak{K}^S_{z_0}(\bar{A})$ if $A \subset K(\bar{A})$ and

$$\frac{K(\alpha, a)}{\alpha(z_0)} \leqslant C \, \|a\|_A \qquad \text{for all} \quad a \in A \text{ and all } \alpha \in S. \tag{2.1}$$

The importance of this notion is seen from the following theorem (see also Remark 2.1).

THEOREM 2.1. *The following conditions are equivalent*:

(i) *A belongs to $\mathfrak{K}^S_{z_0}(\bar{A})$*;

(ii) *$A \subset K(\bar{A})$, and for any Banach space B and any bounded linear operator T: $K(\bar{A}) \to B$ such that*

$$\|Ta\|_B \leqslant M(\gamma) \, \|a\|_\gamma$$

(if $j=1$) for all $\gamma \in \mathbb{T}$ and all $a \in \mathscr{A}$ or (if $j=2$) for almost all $\gamma \in \mathbb{T}$ and all $a \in \mathscr{A}$, $M(\cdot)$ being a bounded (or essentially bounded) function on $\mathbb{T}$, one has

$$\|T\|_{A \to B} \leqslant C D^S_{z_0}(M), \tag{2.2}$$

where C is independent of T.

(iii) *$A \subset K(\bar{A})$, and if T is as in* (ii) *but with $M(\cdot)$ in S, then one has*

$$\|T\|_{A \to B} \leqslant C M(z_0). \tag{2.3}$$

Proof. We give the proof only for $j=2$; the case $j=1$ requires only obvious modifications.

ad (i) $\Rightarrow$ (ii). Assume that (2.1) holds. Let T: $K(\bar{A}) \to B$ be a bounded linear operator such that $\|Ta\|_B \leqslant M(\gamma) \|a\|_\gamma$ a.e. on $\mathbb{T}$ for $a \in \mathscr{A}$. Let $a \in A \subset K(\bar{A})$, choose a number $\varepsilon > 0$ and consider a representation $a = \int_{\mathbb{T}} a(\gamma) \, d\gamma$ such that $a(\cdot) \in \bar{G}$ with

$$\int_{\mathbb{T}} \alpha(\gamma) \, \|a(\gamma)\|_\gamma \, d\gamma \leqslant (1+\varepsilon) \, K_2(\alpha, a).$$

Let us take $a_n(\cdot)\in G$ such that $\|a_n(\gamma)-a(\gamma)\|_{A(\gamma)}\to 0$ a.e., $\|a_n(\gamma)\|_{A(\gamma)}\leqslant C\,\|a(\gamma)\|_{A(\gamma)}$ and set $a_n=\int_{\mathbb{T}} a_n(\gamma)\,d\gamma\in\mathscr{A}$. Choose another arbitrary number $\eta>0$. For all sufficiently big numbers n and m we have

$$\|Ta_n-Ta_m\|_B\leqslant\int_{\mathbb{T}}\|T(a_n(\gamma)-a_m(\gamma))\|_B\,d\gamma$$

$$\leqslant\int_{\mathbb{T}} M(\gamma)\,\|a_n(\gamma)-a_m(\gamma)\|_\gamma\,d\gamma\leqslant\eta.$$

Thus there exists an element $b\in B$ such that $Ta_n\to b$ in B. Since, in addition, a_n converges to a in $K(\bar{A})$ we conclude that $Ta=b$. Hence, for n big enough we have

$$\|Ta\|_B\leqslant\|Ta_n\|_B+\eta\leqslant\int_{\mathbb{T}} M(\gamma)\,\|a_n(\gamma)\|_\gamma\,d\gamma+\eta$$

$$\leqslant\int_{\mathbb{T}} M(\gamma)\,\|a(\gamma)\|_\gamma\,d\gamma+2\eta.$$

Since η is arbitrary, we conclude that $\|Ta\|_B\leqslant\int_{\mathbb{T}} M(\gamma)\,\|a(\gamma)\|_\gamma\,d\gamma$ and, thus, we find

$$\|Ta\|_B\leqslant\int_{\mathbb{T}} M(\gamma)\,\|a(\gamma)\|\,\alpha^{-1}(\gamma)\,\alpha(\gamma)\,d\gamma$$

$$\leqslant\operatorname*{ess\,sup}_\gamma\frac{M(\gamma)}{\alpha(\gamma)}\int_{\mathbb{T}}\alpha(\gamma)\,\|a(\gamma)\|\,d\gamma$$

$$\leqslant(1+\varepsilon)\operatorname*{ess\,sup}_\gamma\frac{M(\gamma)}{\alpha(\gamma)}\,K_2(\alpha,a)$$

$$\leqslant C(1+\varepsilon)\,\|a\|_A\cdot\alpha(z_0)\operatorname*{ess\,sup}_\gamma\frac{M(\gamma)}{\alpha(\gamma)}.$$

Taking the infimum over all $\alpha\in S$ and using that $\varepsilon>0$ is arbitrarily small, this yields

$$\|Ta\|_B\leqslant C\,\|a\|_A\,D^s_{z_0}(M),$$

which means that (2.2) holds.

ad (ii) $\Rightarrow$ (iii). We observe that $D^S_{z_0}(M) \geqslant M(z_0)$ for every M. On the other hand, if $M \in S$ and so $M^{-1} \in S$, we have trivially

$$D^S_{z_0}(M) \leqslant \operatorname*{ess\,sup}_{\gamma} \frac{M(\gamma)\, M^{-1}(\gamma)}{M^{-1}(z_0)} = M(z_0),$$

and the implication follows.

ad (iii) $\Rightarrow$ (i). For any $a \in \mathscr{A}$ we can write $a = a \int_{\mathbb{T}} \varphi(\gamma)\, d\gamma$, where φ is any integrable function such that $\int_{\mathbb{T}} \varphi(\gamma)\, d\gamma = 1$ and, thus, we have

$$K_2(\alpha, a) \leqslant \inf_{\varphi} \int_{\mathbb{T}} \alpha(\gamma)\, \|a\|_\gamma\, \varphi(\gamma)\, d\gamma = \operatorname*{ess\,inf}_{\gamma \in \mathbb{T}} \alpha(\gamma)\, \|a\|_\gamma \leqslant \alpha(\gamma)\, \|a\|_\gamma$$

a.e. on $\mathbb{T}$. Take B to be the vector space $K_2(\bar{A})$ with the norm $K_2(\alpha, \cdot)$. Since $S \subset L^\infty$, we find that $K_2(\bar{A}) \subset B$ and, hence, we can take for T the canonical imbedding I: $K_2(\bar{A}) \to B$. Then

$$\|Ia\|_B = \|a\|_B = K_2(\alpha, a) \leqslant \alpha(\gamma)\, \|a\|_\gamma$$

and therefore, by (2.3) applied with $M = \alpha$, we obtain

$$K_2(\alpha, a) = \|a\|_B = \|Ia\|_B \leqslant C\alpha(z_0)\, \|a\|_A,$$

and the proof is complete. ∎

Now we come to the dual notion.

DEFINITION 2.2. We say that a Banach space B belongs to the *class* $\mathfrak{J}^S_{z_0}(\bar{B})$ if $\Delta \bar{B} \subset B$ and if

$$\|a\|_B \leqslant C \frac{J(\alpha, a)}{\alpha(z_0)} \qquad \text{for all} \quad \alpha \in S. \tag{2.4}$$

The usefulness of this concept is seen from the following theorem (see again Remark 2.1).

THEOREM 2.2. *Let $\mathscr{B}$ be the log-intersection space of the family $\bar{B}$. The following conditions are equivalent*:

(i) *B belongs to $\mathfrak{J}^S_{z_0}(\bar{B})$*;

(ii) *$\|a\|_B \leqslant C D^S_{z_0}(\|a\|_\gamma)$ for all $a \in \mathscr{B}$*;

(iii) $\Delta\bar{B}\subset B$, *and for any Banach space* A *and any bounded linear operator* $T\colon A\to B(\gamma)$ *with* $\|Ta\|_{B(\gamma)}\leqslant M(\gamma)\,\|a\|_A$ *for a.e.* $\gamma\in\mathbb{T}$ *and all* $a\in A$, *where* $M(\gamma)$ *is assumed to be bounded by a positive constant, one has*

$$\|T\|_{A\to B}\leqslant CD^S_{z_0}(M), \tag{2.5}$$

where C *is independent of* T.

(iv) $\Delta\bar{B}\subset B$, *and if* T *is as in* (iii) *but* M *is taken to be in* S, *then one has*

$$\|T\|_{A\to B}\leqslant CM(z_0). \tag{2.6}$$

Proof. ad (i) ⇒ (ii). This implication follows at once by taking the infimum in (2.4) over all $\alpha\in S$.

ad (ii) ⇒ (iii). Let $a\in A$ with $\|a\|_A\leqslant 1$ and write $b=Ta$. Since

$$\operatorname*{ess\,sup}_{\gamma}\|b\|_{B(\gamma)}\leqslant\|M\|_\infty\,\|a\|_A,$$

we have $b\in\Delta\bar{B}$ and, thus, by hypothesis

$$\|Ta\|_B=\|b\|_B\leqslant CD^S_{z_0}(\|b\|_\gamma)\leqslant CD^S_{z_0}(M(\gamma)\,\|a\|_A)\leqslant CD^S_{z_0}(M)\,\|a\|_A$$

establishing (2.5).

ad (iii) ⇒ (iv). This implication is trivial as $D^S_{z_0}(M)=M(z_0)$ whenever $M\in S$.

ad (iv) ⇒ (i). Let us fix $\alpha\in S$ and an element $a\in\Delta\bar{A}$, and let A be the one-dimensional space spanned by a with the norm

$$\|a\|_A=J(\alpha,a)=\operatorname*{ess\,sup}_{\gamma}\{\alpha(\gamma)\,\|a\|_{B(\gamma)}\}.$$

Let I denote the identity operator from A into B. Then we have

$$\begin{aligned}\|I\|_{A\to B(\gamma)}&=\sup_{a\in A}\frac{\|a\|_{B(\gamma)}}{\|a\|_A}\\&\leqslant\frac{\|a\|_{B(\gamma)}}{\alpha(\gamma)\,\|a\|_{B(\gamma)}}\\&=\alpha^{-1}(\gamma)\qquad\text{for}\quad\text{a.e. }\gamma\in\mathbb{T}.\end{aligned}$$

Therefore, using (2.6) with $M(\gamma)=\alpha^{-1}(\gamma)$ we see that

$$\begin{aligned}\|a\|_B = \|Ia\|_B &\leqslant C\alpha^{-1}(z_0)\,\|a\|_A \\ &= C\alpha^{-1}(z_0)\,\operatorname*{ess\,sup}_{\gamma}\,\{\|a\|_{B(\gamma)}\,\alpha(\gamma)\} \\ &= C\,\frac{J(\alpha, a)}{\alpha(z_0)},\end{aligned}$$

which means that $B \in J^S_{z_0}(\bar{B})$. ▮

We have also the following characterizations.

THEOREM 2.3. (a) *$A \in \mathfrak{K}^S_{z_0}(\bar{A})$ if and only if $A \subset (A)^S_{z_0,\,\infty;\,K}$.*
(b) *$A \in \mathrm{J}^S_{z_0}(\bar{A})$ if and only if $(A)^S_{z_0,\,1;\,J} \subset A$.*

Proof. The proof of this theorem requires only obvious modifications of the proof in the classical case of Banach couples (see [1]) so we omit the details. ▮

Remark 2.1. In particular, it follows from Theorem 2.3 that each of the spaces $(A)^S_{z_0,\,\infty;\,K}$ and $(A)^S_{z_0,\,1;\,J}$ belongs both to $\mathfrak{K}^S_{z_0}(\bar{A})$ and to $\mathfrak{J}^S_{z_0}(\bar{A})$. It is likewise easy to check that the complex interpolation space $A[z_0]$ enjoys this property.

2.2. *Relations between the spaces $(A)^{S,\,1}_K$ and $(A)^{S,\,2}_K$*

THEOREM 2.4. *For any bounded interpolation family $\bar{A}$ one has $(A)^{S,\,2}_K \subset (A)^{S,\,1}_K$.*

Proof. It is sufficient to establish that $K_1(\alpha, a) \leqslant K_2(\alpha, a)$ for every $\alpha \in S$ and every $a \in (A)^{S,\,2}_K$.

It follows from the definition of K_2 that for any $\varepsilon > 0$ we can find a function $a(\cdot) \in \bar{G}$ such that $a = \int_{\mathbb{T}} a(\gamma)\,d\gamma$ and $\int \alpha(\gamma)\,\|a(\gamma)\|_\gamma\,d\gamma \leqslant (1+\varepsilon)\,K_2(\alpha, a)$. We also choose a sequence $a_n(\cdot) \in G$ with $a_n(\cdot) \to a(\cdot)$ and write $a_n = \int_{\mathbb{T}} a_n(\gamma)\,d\gamma \in \mathscr{A}$. Let us further introduce the *ad hoc* notation $\Sigma = \sum_\gamma \alpha(\gamma)\,A(\gamma)$. For n and m big enough we have

$$\|a_n - a_m\|_\Sigma \leqslant \int_{\mathbb{T}} \|a_n(\gamma) - a_m(\gamma)\|_\Sigma\,d\gamma \leqslant \int_{\mathbb{T}} \|a_n(\gamma) - a_m(\gamma)\|_{A(\gamma)}\,\alpha(\gamma)\,d\gamma \leqslant \varepsilon.$$

Hence $\{a_n\}_n$ converges to an element b in Σ and, since this space is imbedded in $\mathscr{U}$, we have $b = a$. We conclude that $a \in \Sigma$ and, moreover, that taking n sufficiently large for any $\varepsilon > 0$

$$\|a\|_\Sigma = K_1(\alpha, a) \leqslant \|a_n\|_\gamma + \varepsilon \leqslant \int_{\mathbb{T}} \alpha(\gamma)\ \|a_n(\gamma)\|_\gamma\, d\gamma + \varepsilon$$

$$\leqslant \int_{\mathbb{T}} \alpha(\gamma)\ \|a(\gamma)\|_\gamma\, d\gamma + 2\varepsilon \leqslant (1+\varepsilon)\ K_2(\alpha, a) + 2\varepsilon.$$

Let $\varepsilon \to 0$ and the proof is complete. ∎

Next we note that the methods $(A)_K^{S,1}$ and $(A)_K^{S,2}$ do not coincide in general. For example, if S consists of the one function $\alpha \equiv 1$, then we have $(A)_K^{S,1} = \sum_\gamma A(\gamma)$ and if $B(\gamma) = A(\gamma)$ a.e. on $\mathbb{T}$, while $B(\gamma) \neq A(\gamma)$ on a set of measure zero, then in general $(B)_K^{S,1} = \sum_\gamma B(\gamma) \neq (A)_K^{S,1}$ but $(B)_K^{S,2} = (A)_K^{S,2}$.

However, the situation changes if we restrict ourselves to *countable* families.

THEOREM 2.5. *If $\bar{A}$ is countable (i.e. if $A(\gamma) = A_n$ for all $\gamma \in \Gamma_n$, where $\{\Gamma_n\}_{n \in \mathbb{N}}$ is a* partition *into intervals of $\mathbb{T}$) such that $\mathscr{A}$ is dense in $A(\gamma)$ for every $\gamma \in \mathbb{T}$ and all functions $\alpha \in S$ are regular*[1], *then $(A)_K^{S,1} = (A)_K^{S,2}$.*

Proof. According to Theorem 2.4 it is obviously sufficient to prove that $K_2(\alpha, a) \leqslant K_1(\alpha, a)$ for every $\alpha \in S$ and $a \in (A)_K^{S,1}$. By the definition of K_1 we can find, for any given $\varepsilon > 0$, elements $a_n \in A_n$ such that $a = \sum_n a_n$ and

$$\sum_n \inf_{\Gamma_n} \alpha(\gamma)\ \|a_n\|_{A_n} \leqslant (1+\varepsilon)\ K_1(\alpha, a).$$

Let us take

$$a(\gamma) = \sum_n a_n \varphi_n(\gamma)\ \chi_{\Gamma_n}(\gamma)$$

with

$$\int_{\Gamma_n} \varphi_n(\gamma)\ d\gamma = 1$$

and

$$\int_{\Gamma_n} \alpha(\gamma)\ \varphi_n(\gamma)\ d\gamma \leqslant (1+\varepsilon) \operatorname*{ess\,inf}_{\Gamma_n} \alpha(\gamma) = (1+\varepsilon) \inf_{\Gamma_n} \alpha(\gamma),$$

[1] An integrable function is said to be *regular* if every point is a Lebesgue point.

where the last equality holds since α is regular by hypothesis. Using the density assumption we find that $a(\cdot) \in \bar{G}$ and

$$K_2(\alpha, a) \leqslant \int_{\mathbb{T}} \alpha(\gamma) \, \|a(\gamma)\|_\gamma \, d\gamma = \sum_n \|a_n\|_{A_n} \int_{\Gamma_n} \alpha(\gamma) \, \varphi_n(\gamma) \, d\gamma$$

$$\leqslant (1+\varepsilon) \sum_n \|a_n\|_{A_n} \inf_{\Gamma_n} \alpha(\gamma) \leqslant (1+\varepsilon)^2 \, K_1(\alpha, a).$$

Letting $\varepsilon \to 0$ completes the proof. ∎

Remark 2.2. Each of the interpolation spaces $(A)_K^{S,1}$ and $(A)_K^{S,2}$ constitutes a generalization of the constructions of the Cobos–Peetre [7], Fernández [13] and Sparr [24].

3. NEW LIMIT CONSTRUCTIONS

We consider a finite collection of consecutive points $\Pi = \{(x_j, y_j)\}_j$ on the unit circumference and let z_0 denote a point in the unit disk D. Moreover, Γ_j denotes the arc between (x_j, y_j) and (x_{j+1}, y_{j+1}), while $|\Gamma_j|_{z_0} = \int_{\Gamma_j} P_{z_0}(t) \, dt$ stands for the harmonic measure of Γ_j with respect to z_0; as before $P_{z_0}(t)$ is the Poisson kernel.

Now we construct two subsets S_{p} and S_{d} of the space $L^\infty(\mathbb{T})$ as follows:[2] We first select a sequence Π_N $(N = 1, 2, ...)$ of finite collections of points with $\Pi_N \subset \Pi_{N+1}$. Letting $\{\Gamma_j^N\}_j$ denote the partition of $\mathbb{T}$ by the points of Π_N, we assume also that $\max_j |\Gamma_j^N| \to 0$ as $N \to \infty$. For each index N we let S_{p}^N be the set of all functions $\alpha = \alpha_{n,m}$ such that

$$\alpha_{n,m}(\gamma) = 2^{n x_j^N + m y_j^N} \qquad \text{for} \quad \gamma \in \Gamma_j^N$$

where n and m are arbitrary integers and, similarly, we let S_{d}^N be the set of all functions $\alpha = \alpha_{\mathbf{n}}$ such that

$$\alpha_{\mathbf{n}}(\gamma) = 2^{n_j} \qquad \text{for} \quad \gamma \in \Gamma_j^N$$

where $\mathbf{n} = \{n_j\}_j$ is an arbitrary collection of integers. Finally, we put

$$S_{\mathrm{p}} = \bigcup_N S_{\mathrm{p}}^N \qquad \text{and} \qquad S_{\mathrm{d}} = \bigcup_N S_{\mathrm{d}}^N.$$

[2] The subscripts p and d are standing for “polynomial” and “dyadic” respectively.

We also introduce the set S_{T}[3] obtained as the pointwise limit of the functions $\alpha_{n,m} = \alpha^N_{n,m}$ as $N \to \infty$, that is, S_{T} consists of all functions $\alpha_{n,m}$ of the form

$$\alpha_{n,m}(\gamma) = 2^{n \cos \gamma + m \sin \gamma} \qquad \text{for} \quad \gamma \in \mathbb{T}.$$

The corresponding interpolation spaces were introduced in [20, 23] and further studied in [22].

We remark that, as is readily seen, both $(A)^{S_{\mathrm{T}}}_J$ and $(A)^{S_{\mathrm{T}}}_K$ are Banach spaces continuously imbedded in $\mathscr{U}$ with $(A)^{S_{\mathrm{T}}}_J \subset (A)^{S_{\mathrm{T}}}_K$. These statements follow from the existence of a compact set K such that (see [3])

$$\sum_{\alpha \in S_{\mathrm{T}}} \frac{\inf_K \alpha(z)}{\alpha(z_0)} < \infty.$$

In fact, taking K to be a circle with center at the origin $(0, 0)$ and radius $r > (\alpha^2 + \beta^2)^{1/2}$ we find that

$$\inf_K \alpha(z) = \inf_{\gamma \in [0, 2\pi)} 2^{nr \cos \gamma + mr \sin \gamma} = 2^{-r\sqrt{n^2+m^2}}$$

and, hence,

$$\sum_{n,m} \frac{2^{-r\sqrt{n^2+m^2}}}{2^{n\alpha + m\beta}} \leqslant \sum_{n,m} 2^{-\sqrt{n^2+m^2}(r - \sqrt{\alpha^2+\beta^2})} < \infty. \qquad \blacksquare$$

In this context we also require the following notation: Let $\{\Gamma_j\}_j$ be any partition of $\mathbb{T}$. Then we set

$$\Delta_j(\bar{A}) = \left\{ a \in \bigcap_{\gamma \in \Gamma_j} A(\gamma) \colon \operatorname*{ess\,sup}_{\Gamma_j} \|a\|_\gamma < \infty \right\};$$

$$K_j(\bar{A}) = \{ a \in \mathscr{U} \colon K_j(1, a) < \infty \},$$

where either (discrete case)

$$K_{1,j}(1, a) = \inf \left\{ \sum_i \|a_i\|_{\gamma_i} \colon a = \sum_i a_i, \{\gamma_i\} \subset \Gamma_j \right\}$$

or (continuous case)

$$K_{2,j}(1, a) = \inf \left\{ \int_{\Gamma_j} \|a(\gamma)\|_\gamma \, d\gamma \colon a = \int_{\Gamma_j} a(\gamma) \, d\gamma \right\}.$$

[3] The letter T is for "trigonometric".

Moreover, the Cobos–Peetre J- and K-spaces corresponding to $\Pi_N = \{(x_j^N, y_j^N)\}$ will be denoted $(A_1, A_2, ..., A_N)_{(\alpha, \beta), p; J}$ and $(A_1, A_2, ..., A_N)_{(\alpha, \beta), p; K}$ respectively (see [24]), while the analogous Sparr J- and K-spaces will be denoted $(A_1, A_2, ..., A_N)_{\theta, p; J}$ and $(A_1, A_2, ..., A_N)_{\theta, p; K}$ with $\theta = (\theta_1, \theta_2, ..., \theta_N)$ respectively (see [24]).

The following relations hold:

THEOREM 3.1. *Let $\bar{A}$ be an interpolation family and let*

$$(\alpha, \beta) = \sum_j |\Gamma_j|_{z_0} (x_j, y_j)$$

be a point in D, and set $\Gamma_j = \Gamma_j^N$ and let $\theta = \bar{\theta} = (|\Gamma_2|_{z_0}, |\Gamma_3|_{z_0}, ..., |\Gamma_N|_{z_0})$. Then

(i) $(A)^{S_{\mathrm{p}}^N}_{z_0, p; J} = (\Delta_1(\bar{A}), \Delta_2(\bar{A}), ..., \Delta_N(\bar{A}))_{(\alpha, \beta), p; J}$;

(ii) $(A)^{S_{\mathrm{d}}^N}_{z_0, p; J} = (\Delta_1(\bar{A}), \Delta_2(\bar{A}), ..., \Delta_N(\bar{A}))_{\theta, p; J}$;

(iii) $(A)^{S_{\mathrm{p}}^N}_{z_0, p; K} = (K_1(\bar{A}), K_2(\bar{A}), .., K_N(\bar{A}))_{(\alpha, \beta), p; K}$;

(iv) $(A)^{S_{\mathrm{d}}^N}_{z_0, p; K} = (K_1(\bar{A}), K_2(\bar{A}), ..., K_N(\bar{A}))_{\theta, p; K}$.

Proof. We only prove (i) and (iv) because the proofs of (ii) and (iii) are quite similar.

ad (i). Consider $a \in \mathscr{A} = \Delta(\bar{A}) = \bigcap_j \Delta_j(\bar{A})$, fix $n, m \in \mathbb{Z}_+$ and let $\alpha_{n, m}(\gamma) = 2^{nx_j + my_j}$ for $\gamma \in \Gamma_j$. Then

$$J(\alpha_{n, m}, a) = \sup_j 2^{nx_j + my_j} \operatorname*{ess\,sup}_{\Gamma_j} \|a\|_\gamma = \max_j 2^{nx_j + my_j} \|a\|_{\Delta_j(A)} = J(2^n, 2^m; a),$$

where $J(2^n, 2^m; \cdot)$ is the Cobos–Peetre J-functional for the N-tuple $(\Delta_1(\bar{A}), \Delta_2(\bar{A}), ..., \Delta_N(\bar{A}))$ (see [7]). The proof of (i) follows from this relation and the observation that $\sum_j (nx_j + my_j) |\Gamma_j|_{z_0} = n\alpha + m\beta$ so that

$$\alpha_{n, m}(z_0) = 2^{n\alpha + m\beta}.$$

ad (iv). Consider the function $\alpha = \alpha_{\mathbf{n}}$ in S_d^N, where

$$\alpha(\gamma) = \alpha_{\mathbf{n}}(\gamma) = 2^{n_j} \qquad \text{for} \quad \gamma \in \Gamma_j.$$

Then

$$K(\alpha, a) = \inf \left\{ \int_{\mathbb{T}} \alpha(\gamma) \|a(\gamma)\|_\gamma \, d\gamma \colon a = \int_{\mathbb{T}} a(\gamma) \, d\gamma \right\}$$

$$= \inf \left\{ \sum_j 2^{n_j} \int_{\Gamma_j} \|a(\gamma)\|_\gamma \, d\gamma \colon a = \int_{\mathbb{T}} a(\gamma) \, d\gamma \right\}. \qquad (3.1)$$

Now we write $a_j = \int_{\Gamma_j} a(\gamma)\, d\gamma$ and observe that $a_j \in K_j(A)$. Therefore using (3.1) we easily find that

$$K(\alpha, a) = \inf_{a = a_1 + \cdots + a_N} \sum_j 2^{n_j} \|a_j\|_{K_j(A)} = K(2^{\mathbf{n}}, a),$$

where $K(2^{\mathbf{n}}, \cdot)$ is the Sparr K-functional (see [24]), and the proof of (iv) follows. ∎

Remark 3.1. According to [6, Theorem 3.1] we have (for every N) the inclusions

$$(A)_J^{S_{\mathrm{p}}^N} \subset (A)_J^{S_{\mathrm{d}}^N} \subset (A)_K^{S_{\mathrm{d}}^N} \subset (A)_K^{S_{\mathrm{p}}^N}.$$

3.1. *The "dyadic" or Sparr Limit Case*

THEOREM 3.2. *The space $(A)_K^{S_{\mathrm{d}}}$ consists of all elements $a \in \bigcap_N (A)_K^{S_{\mathrm{d}}^N}$ such that*

$$\|a; (A)_K^{S_{\mathrm{d}}}\| = \sup_N \|a; (A)_K^{S_{\mathrm{d}}^N}\| < \infty.$$

Proof. Since $S_{\mathrm{d}}^N \subset S_{\mathrm{d}}^{N+1} \subset \cdots \subset S_{\mathrm{d}}$, we have $(A)_K^{S_{\mathrm{d}}} \subset (A)_K^{S_{\mathrm{d}}^N}$ for all N. Therefore we obtain

$$\|a; (A)_K^{S_{\mathrm{d}}}\| = \left(\sum_{\alpha \in S_{\mathrm{d}}} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p}$$
$$= \lim_{N \to \infty} \left(\sum_{\alpha \in S_{\mathrm{d}}^N} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} = \sup_N \|a; (A)_K^{S_{\mathrm{d}}^N}\|. \quad ∎$$

Remark 3.2. The "upper" space $U_M(A, Z)$ of Cwikel–Janson [11], with M equal to the Sparr K-method with parameter p, is obviously imbedded in the space $(A)_{z_0, p; K}^{S_{\mathrm{d}}, 1}$, because in the definition of [11] the authors take the intersection over all partitions of $\mathbb{T}$, while in our case the intersection is taken over a certain fixed collection of partitions. We do not know if there exists a similar imbedding with the space $(A)_{z_0, p; K}^{S_{\mathrm{d}}, 2}$, that is, when we use the continuous K_2-functional instead of the discrete K_1-functional.

Next we state a result for the J-functional which is closely related to (2.18) in [11].

THEOREM 3.3. *For every $a \in (A)_{z_0, 1; J}^{S_{\mathrm{d}}}$ there exists a sequence (a_N), where a_N in $(A)_{z_0, 1; J}^{S_{\mathrm{d}}^N}$, such that $\|a - a_N; (A)_{z_0, 1; J}^{S_{\mathrm{d}}}\| \to 0$ as $N \to \infty$ and, moreover,*

$$\|a; (A)_{z_0, 1; J}^{S_{\mathrm{d}}}\| = \lim_{N \to \infty} \|a_N; (A)_{z_0, 1; J}^{S_{\mathrm{d}}^N}\|.$$

Proof. If $a \in (A)^{S_d}_{z_0, 1; J}$, then we can write $a = \sum_S a_\alpha$ where the convergence is uniform in $\mathscr{U}$, since

$$\sum_{\alpha \in S} \|a_\alpha\|_{\mathscr{U}} \leqslant \sum_{\alpha \in S} \inf_{\gamma} \|a_\alpha\|_\gamma \leqslant \sum_{\alpha \in S} \frac{J(\alpha, a_\alpha)}{\alpha(z_0)} < \infty.$$

Therefore we can choose $a_N = \sum_{\alpha \in S^N_d} a_\alpha \in (A)^{S^N_d}_{z_0, 1; J}$, in which case we find

$$\|a - a_N; (A)^{S_d}_{z_0, 1; J}\| \leqslant \sum_{a \in S_d \setminus S^N_d} \frac{J(\alpha, a_\alpha)}{\alpha(z_0)} \to 0 \qquad \text{as} \quad N \to \infty.$$

We also have, for any $\varepsilon > 0$ and with a suitable choice of the a_α,

$$\begin{aligned}(1+\varepsilon)\,\|a; (A)^{S_d}_{z_0, 1; J}\| &\geqslant \sum_{a \in S_d} \frac{J(\alpha, a_\alpha)}{\alpha(z_0)} = \overline{\lim_{N \to \infty}} \sum_{a \in S^N_d} \frac{J(\alpha, a_\alpha)}{\alpha(z_0)} \\ &\geqslant \overline{\lim_{N \to \infty}}\ \|a_N; (A)^{S^N_d}_{z_0, 1; J}\|.\end{aligned}$$

On the other hand, we find

$$\|a; (A)^{S_d}_{z_0, 1; J}\| = \lim_{N \to \infty} \|a_N; (A)^{S_d}_{z_0, 1; J}\| \leqslant \underline{\lim}_{N \to \infty} \|a_N; (A)^{S^N_d}_{z_0, 1; J}\|$$

We conclude that

$$\|a; (A)^{S_d}_{z_0, 1; J}\| = \lim_{N \to \infty} \|a_N; (A)^{S^N_d}_{z_0, 1; J}\|. \quad \blacksquare$$

Remark 3.3. We observe that the interpolation space $(A)^{S_d}_{z_0, 1; J}$ is continuously imbedded in the Cwikel–Janson "lower" space $L_M(A, Z)$ where M is the Sparr J-method with parameter 1.

3.2. *The "Polygonal" or Cobos–Peetre Case*

In this case we do not have any monotonicity property similar to the one in Sparr's case. The situation is more complicated and, as our next theorem shows, we have even $(A)^{S_p}_K = 0$ except when $p = \infty$.

THEOREM 3.4. *Let* $\tilde{S}^1_p = S^1_p$ *and* $\tilde{S}^N_p = S^N_p \setminus \{1\}$ $(N = 2, 3, \ldots)$. *Then the space* $(A)^{S_p}_{z_0, p; K}$ *consists of all elements* $a \in \bigcap_N (A)^{\tilde{S}^N}_{z_0, p; K}$ *such that*

$$\left(\sum_N \|a; A^{\tilde{S}^N_p}_{z_0, p; K}\|^p\right)^{1/p} < \infty.$$

Proof. Let $\alpha \in \tilde{S}^N_{\mathrm{p}} \cap \tilde{S}^{N'}_{\mathrm{p}}$ with $N < N'$. Consider the collection of consecutive points $\{(x^{N'}_k, y^{N'}_k)\}$ at step N' on the arc Γ^N_j. Then we have

$$nx^{N'}_k + my^{N'}_k = nx^{N'}_{k+1} + my^{N'}_{k+1} \text{ implying that } \frac{y^{N'}_k - y^{N'}_{k+1}}{x^{N'}_k - x^{N'}_{k+1}} = -\frac{n}{m}.$$

This means that the sign of the last ratio is constant, and this is of course not possible as soon as we have at least three points in each quadrant. Therefore we must have $\tilde{S}^N_{\mathrm{p}} \cap \tilde{S}^{N'}_{\mathrm{p}} = \emptyset$ if $N < N'$ and N is big enough. Hence[4]

$$\begin{aligned} \|a; (A)^{S_{\mathrm{p}}}_{z_0, p; K}\| &= \left(\sum_{\alpha \in S_{\mathrm{p}}} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} \\ &\approx \left(\sum_N \sum_{\alpha \in \tilde{S}^N_{\mathrm{p}}} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} = \left(\sum_N \|a; (A)^{\tilde{S}^N_{\mathrm{p}}}_{z_0, p; K}\|^p \right)^{1/p}. \quad \blacksquare \end{aligned}$$

Since $A(\gamma) \subset \mathcal{U}$ with norm 1 we have $(A)^{S^N_{\mathrm{p}}}_K \subset \mathcal{U}$, likewise with norm 1. It follows from Theorem 3.4 that $(A)^{S_{\mathrm{p}}}_{z_0, p; K} = 0$ except for $p = \infty$. This suggests that we replace S_{p} by a certain subset $\hat{S}_{\mathrm{p}}$ so that we can guarantee that at least $(A)^{S_{\mathrm{p}}}_{z_0, p; K} \neq 0$. We must avoid the situation that the set $\hat{S}_{\mathrm{p}}$ contains two functions which are equivalent in the sense of the following definition.

DEFINITION 3.1. Two positive functions α and β defined on $\mathbb{T}$ are said to be *C-equivalent*, where C is a positive number, if

$$C^{-1}\beta(\gamma) \leqslant \alpha(\gamma) \leqslant C\beta(\gamma) \qquad \text{for} \quad \gamma \in \mathbb{T},$$

which we agree to write as $\alpha \approx^C \beta$. More generally, if for every $\alpha \in S$ there exists $\alpha' \in S'$ such that $\alpha \approx^C \alpha'$ we shall use the notation $S \subset^C S'$. In this situation the correspondence $\alpha \xrightarrow{\Phi} \alpha'$ is *almost injective* in the sense that there exists an integer $K > 0$ such that, for every α', $\operatorname{card}\{\alpha : \Phi(\alpha) = \alpha'\} \leqslant K$.

Let $\{\alpha_1, \alpha_2, ...\}$ be the set S_{p} enumerated in some way. We construct the subset $\hat{S}_{\mathrm{p}}$ in the following way: We keep α_1; if $\alpha_2 \approx^2 \alpha_1$ we discard it but if $\alpha_2 \not\approx^2 \alpha_1$ we keep it; quite generally, proceeding inductively for $N \geqslant 2$, if $\alpha_N \approx^2 \alpha_j$ for some $j < N$ we discard it but if not we keep it.

Now we are ready to formulate our main result for the K-method.

THEOREM 3.5. *We have* $A^{S_{\mathrm{d}}}_K \subset A^{\hat{S}_{\mathrm{p}}}_K \subset A^{S_T}_K$.

[4] The notation $x \approx y$ means, here and in the sequel, that there exists a constant C (independent of "everything") such that $C^{-1}x \leqslant y \leqslant C_2\, y$; cf. Definition 3.1 below.

Proof. In order to prove the second inclusion we assume for a moment that

$$S_{\mathrm{T}} \subset^4 \hat{S}_{\mathrm{p}} \tag{3.2}$$

Let $a \in (A)_K^{\hat{S}_{\mathrm{p}}}$ and $\beta \in S_{\mathrm{T}}$. Then there exists $\alpha \in \hat{S}_{\mathrm{p}}$ such that $\beta \approx^4 \alpha$ and, hence,

$$\frac{K(\beta, a)}{\beta(z_0)} \leqslant C \frac{K(\alpha, a)}{\alpha(z_0)}$$

for some constant C independent of β and α. Moreover, since $\alpha \in \hat{S}_{\mathrm{p}}$ there are only a finite number, say, K functions $\beta \in S_{\mathrm{T}}$ such that $\alpha \approx^4 \beta$. It follows that

$$\begin{aligned} \|a; (A)_K^{S_{\mathrm{T}}}\| &= \left(\sum_{\beta \in S_{\mathrm{T}}} \left(\frac{K(\beta, a)}{\beta(z_0)} \right)^p \right)^{1/p} \\ &\leqslant CK \left(\sum_{\alpha \in \hat{S}_{\mathrm{p}}} \left(\frac{K(\alpha, a)}{\alpha(z_0)} \right)^p \right)^{1/p} = C_0 \, \|a; (A)_K^{\hat{S}_{\mathrm{p}}}\|. \end{aligned}$$

Thus the second inclusion holds in the hypothesis (3.2).

Now we prove (3.2). Let $\alpha \in S_{\mathrm{T}}$. Then there exist integers n and m such that $\alpha(\gamma) = \alpha_{n,m}(\gamma) = 2^{n \cos \gamma + m \sin \gamma}$. Now since $c_{n,m} = \inf 2^{n \cos \gamma + m \sin \gamma} \neq 0$ there exists an integer $N = N(n, m)$ such that

$$|2^{n \cos \gamma + m \sin \gamma} - 2^{nx_j^N + my_j^N}| < c_{m,n}$$

for every $\gamma \in \Gamma_j^N$ and every j. It follows that

$$\tfrac{1}{2} \cdot 2^{nx_j^N + my_j^N} \leqslant 2^{n \cos \gamma + m \sin \gamma} \leqslant 2 \cdot 2^{nx_j^N + my_j^N}$$

for every $\gamma \in \Gamma_j^N$. Therefore, if we define a function $\alpha_{n,m}^N$ such that $\alpha_{n,m}^N(\gamma) = 2^{nx_j^N + my_j^N}$ for $\gamma \in \Gamma_j^N$ it follows that $\alpha_{n,m}^N \approx^2 \alpha_{n,m}$.

If $\alpha_{n,m}^N \in \hat{S}_{\mathrm{p}}$ we are done so let us assume that $\alpha_{n,m}^N \notin \hat{S}_{\mathrm{p}}$. Then there exists $\beta \in \hat{S}_{\mathrm{p}}$ such that $\beta \approx^2 \alpha_{n,m}^N$ and, hence, $\alpha_{n,m} \approx^4 \beta$. We conclude that for all $\alpha \in S_{\mathrm{T}}$ there exists $\beta \in \hat{S}_{\mathrm{p}}$ such that $\alpha \approx^4 \beta$. Next we note that if $\alpha_{n,m} \approx^4 \beta$ and $\alpha_{n',m'} \approx^4 \beta$, then $\alpha_{n,m} \approx^{16} \alpha_{n',m'}$ and, hence,

$$\frac{1}{16} < \frac{2^{n \cos \gamma + m \sin \gamma}}{2^{n' \cos \gamma + m' \sin \gamma}} < 16.$$

Thus $-4 \leqslant (n-n')\cos\gamma + (m-m')\sin\gamma \leqslant 4$ and it follows that there exists an integer $K>0$ such that, for all n, m,

$$\text{card}\{(n', m') \colon \alpha_{n', m'} \approx^4 \beta\} \leqslant K,$$

i.e. we have also almost injectivity. This proves (3.2) and so the second inclusion in the theorem.

In order to prove the first inclusion we consider the functions

$$\alpha^N_{n,m} = \sum_j 2^{nx_j + my_j} \chi_{\Gamma^N_j} \qquad \text{and} \qquad \tilde{\alpha}^N_{n,m} = \sum_j 2^{n(x_j - x_1) + m(y_j - y_1)} \chi_{\Gamma^N_j}$$

and the set $\tilde{S}_{\mathrm{p}} = \{\tilde{\alpha} \colon \alpha \in \hat{S}_{\mathrm{p}}\}$. We note that

$$\frac{K(\alpha^N_{n,m}, a)}{2^{n\alpha + m\beta}} = \frac{K(\tilde{\alpha}^N_{n,m}, a)}{2^{n(\alpha - x_1) + m(\beta - y_1)}}.$$

It follows that $(A)^{\hat{S}_{\mathrm{p}}}_{z_0, p; K} = (A)^{\tilde{S}_{\mathrm{p}}}_{z_0, p; K}$. Moreover, one has

$$\begin{aligned}&\text{for every } \alpha \in \tilde{S}_{\mathrm{p}} \text{ there exists a unique element}\\ &\beta \in S_{\mathrm{p}} \text{ such that } \alpha \approx^2 \beta \text{ and } \alpha(z_0) \approx \beta(z_0).\end{aligned} \tag{3.3}$$

Using (3.3) one finds

$$(A)^{S_{\mathrm{p}}}_{z_0, p; K} \subset (A)^{\tilde{S}_{\mathrm{p}}}_{z_0, p; K} \subset (A)^{\hat{S}_{\mathrm{p}}}_{z_0, p; K}.$$

In order to prove (3.3) let $\alpha \in \tilde{S}_{\mathrm{p}}$. Then there exist n, m such that

$$\alpha(\gamma) = \sum_j 2^{n(x^N_j - x_1) + m(y^N_j - y_1)} \chi_{\Gamma^N_j} \approx^2 \sum_j 2^{n_j} \chi_{\Gamma^N_j} = \beta(\gamma) \in S_{\mathrm{d}},$$

where $n_j = [n(x^N_j - x_1) + m(y^N_j - y_1)]$ (integer part). Moreover, we have

$$\alpha(z_0) = 2^{n(\alpha - x_1) + m(\beta - y_1)} = 2^{\sum_j (n(x^N_j - x_1) + m(y^N_j - y_1))} \approx 2^{\sum_j n_j |\Gamma^N_j|_{z_0}} = \beta(z_0).$$

It remains to prove that the correspondence $\alpha \mapsto \beta$ is injective.

Assume that, on the contrary, we have two different functions $\alpha^N_{n,m}$ and $\alpha^{N'}_{n',m'}$, with $N \leqslant N'$, such that

$$[n(x^N_j - x_1) + m(y^N_j - y_1)] = [n'(x^{N'}_j - x_1) + m'(y^{N'}_j - y_1)]$$

whenever the two points $\{(x^N_j, y^N_j)\}$ and $\{(x^{N'}_j, y^{N'}_j)\}$ are on the same arc Γ^N_j. Then we have

$$-1 < (n(x^N_j - x_1) + m(y^N_j - y_1)) - (n'(x^{N'}_j - x_1) + m'(y^{N'}_j - y_1)) < 1,$$

that is,

$$\frac{1}{2} < \frac{2^{n(x_j^N - x_1) + m(y_j^N - y_1)}}{2^{n'(x_j^{N'} - x_1) + m'(y_j^{N'} - y_1)}} < 2.$$

Therefore $\alpha_{n,m}^N \approx^2 \alpha_{n',m'}^{N'}$, and thus, by the construction of the set $\hat{S}_{\mathrm{p}}$, we have $\alpha_{n,m}^N = \alpha_{n',m'}^{N'}$. This contradiction completes the proof. ▮

Let now $\hat{S}_{\mathrm{T}}$ be a subset of S_{T} such that the relation $\alpha_{n,m}^N \approx^{16} \alpha_{n',m'}^{N'}$ does not hold for any $(n', m') \neq (n, m)$. Now we can formulate the corresponding result for the J-method.

THEOREM 3.6. *We have* $(A)_J^{\hat{S}_{\mathrm{T}}} \subset (A)_J^{\hat{S}_{\mathrm{p}}} \subset (A)_J^{S_{\mathrm{d}}}$.

Proof. For the proof of the first inclusion we begin by noting that it is easily seen that $\hat{S}_{\mathrm{T}} \subset^4 \hat{S}_{\mathrm{p}}$. Therefore if $\alpha_{n,m} \approx^4 \beta$ and $\alpha_{n',m'} \approx^4 \beta$, then $\alpha_{n',m'} \approx^{16} \alpha_{n,m}$ and we conclude that only one of these two functions can be in $\hat{S}_{\mathrm{T}}$. In other words, the correspondence $\alpha \mapsto \alpha'$ is injective. Now, assume that $a \in (A)_J^{\hat{S}_{\mathrm{T}}}$, which means that we can write

$$a = \sum_{\alpha \in \hat{S}_{\mathrm{T}}} a_\alpha \qquad \text{with} \quad \left(\sum_\alpha \left(\frac{J(\alpha, a_\alpha)}{\alpha(z_0)} \right)^p \right)^{1/p} \leqslant (1+\varepsilon) \, \|a; (A)_J^{\hat{S}_{\mathrm{T}}}\|.$$

Moreover, if $\alpha \in \hat{S}_{\mathrm{T}}$, then there exists $\beta \in \hat{S}_{\mathrm{p}}$ such that $\alpha \approx^4 \beta$ and, hence, if we write $b_\beta = a_\alpha$, then $a = \sum b_\beta$ with β running through $\hat{S}_{\mathrm{p}}$ and

$$\left(\sum_\beta \left(\frac{J(\beta, b_\beta)}{\beta(z_0)} \right)^p \right)^{1/p} \leqslant C \left(\sum_\alpha \left(\frac{J(\alpha, a_\alpha)}{\alpha(z_0)} \right)^p \right)^{1/p} \leqslant (1+\varepsilon) \, \|a; (A)_J^{\hat{S}_{\mathrm{T}}}\|.$$

The proof of the second inclusion follows by using (3.3) and similar arguments as those used in the proof of the first embedding in Theorem 3.5 so we omit the details. ▮

Remark 3.4. The reason why we have to use $\hat{S}_{\mathrm{T}}$ (instead of S_{T}) is that we do not have uniform convergence $\mathscr{U}$ in the sum $a = \sum a_\alpha$ and therefore we cannot reorder its elements. However, if $p = 1$ this can be done. Moreover, considering finite sums, we can prove, assuming that $(A)_J^{\hat{S}_{\mathrm{p}}}$ is a Banach space, that the first imbedding in Theorem 3.6 holds also with $\hat{S}_{\mathrm{T}}$ replaced by S_{T}.

Remark 3.5. The spaces $(A)_K^{\hat{S}_{\mathrm{T}}}$ and $(A)_K^{S_{\mathrm{T}}}$ coincide up to equivalence of norm.

We close this section with the following *summary* of the imbeddings obtained:

$$A_K^{S_{\rm d}} \subset A_K^{\hat{S}_{\rm p}} \subset A_K^{S_T}; \qquad (A)_J^{\hat{S}_{\rm T}} \subset (A)_J^{\tilde{S}_{\rm p}} \subset (A)_J^{S_{\rm d}};$$

$U_M(A, Z) \subset (A)_K^{S_{\rm d},\,1}$ where M is the Sparr K-method with parameter p;

$(A)_{z_0,\,1,\,J}^{S_{\rm d}} \subset L_M(A, Z)$ where M is the Sparr J-method with parameter 1.

4. CALCULATION OF THE FUNCTION $D_{z_0}^S$

4.1. *The Dyadic Case* $S = S_{\rm d}$

According to [3] we have the exact result

$$D_{z_0}^{S_{\rm d}}(M) = M(z_0).$$

4.2. *The Polynomial Case* $S = \hat{S}_{\rm p}$

Since $\hat{S}_{\rm p}$ is not a group, it is not known, in general, how to get a good estimate of the norm of the interpolated operator. However, in some cases (for example, if $p = \infty$) it is known that

$$D_{z_0}^{\hat{S}_{\rm p}}(M) \leqslant \sup_N D_{z_0}^{S_{\rm p}^N}(M),$$

and in this case $D_{z_0}^{S_{\rm p}^N}(M)$ represents the norm of the interpolated operator.

We will now calculate $D_{z_0}^{S_{\rm p}^N}(M)$. To this end we first introduce some notation.

Let (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) be points on the unit circumference $\mathbb{T}$ and let (α, β) be a point of the unit disk D. By Γ_j we denote the arc between (x_j, y_j) and (x_{j+1}, y_{j+1}) $(j = 1, 2, ..., n-1)$. Furthermore, if (α, β) lies inside the triangle with vertices at the points (ξ_1, η_1), (ξ_2, η_2) and (ξ_3, η_3) we say that the numbers λ_1, λ_2 and λ_3 are the *barycentric coordinates* of (α, β) with respect to this triangle if they constitute the (unique) solution of the linear system

$$\begin{cases} \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda_3 \xi_3 = \alpha, \\ \lambda_1 \eta_1 + \lambda_2 \eta_2 + \lambda_3 \eta_3 = \beta, \\ \lambda_1 + \lambda_2 + \lambda_3 = 1. \end{cases}$$

It is well-known that all λ_j are > 0.

The following result is contained in [6] but the present proof is different.

THEOREM 4.1. *Let* $M_j = \sup_{\Gamma_j} M(\gamma)$ *and let* $c_1 = c_1(i, j, k)$, $c_2 = c_2(i, j, k)$ *and* $c_3 = c_3(i, j, k)$ *be the barycentric coordinates of* z_0 *with respect to the triangle with vertices* $\{(x_i, y_i), (x_j, y_j), (x_k, y_k)\}$ $(i < j < k)$. *Then*

$$D_{z_0}^{S_N}(M) \approx \max\{M_i^{c_1(i,j,k)}, M_j^{c_2(i,j,k)}, M_k^{c_3(i,j,k)}\},$$

where the maximum is taken over all triangles containing z_0.

Remark 4.1. Let z_0 be inside the polygon Π with vertices (x_1, y_1), (x_2, y_2), ..., (x_n, y_n). It is easy to see that $D_{z_0}^{S_{\mathrm{P}}^N}(M)$ coincides with the quantity

$$D_{\alpha,\beta} = D_{\alpha,\beta}(M_1, M_2, ..., M_n) \approx \inf_{t, s \geqslant 0} \{\max_{1 \leqslant j \leqslant n} M_j t^{x_j - \alpha} s^{y_j - \beta}\},$$

where $(\alpha, \beta) = \sum_j |\Gamma_j|_{z_0} (x_j, y_j)$. Therefore Theorem 4.1 is more or less identical with Theorem 1.8 in [6], but the present proof, based on the Legendre transform is different.

Proof. Writing $M_j = e^{L_j}$, $t = e^{\xi}$ and $s = e^{\eta}$ we have to compute the expression

$$L^{\#}(\alpha, \beta) = \inf_{\xi, \eta \in \mathbb{R}} \{\max_{1 \leqslant j \leqslant n} \{\xi(x_j - \alpha) + \eta(y_j - \beta) + L_j\}\}.$$

Let Γ_Π denote the boundary of the polygon Π and define a function $L: \Gamma_\Pi \to \mathbb{R}^+$ as follows: $L(x, y) = \lambda L_j + \mu L_{j+1}$ if $(x, y) = \lambda(x_j, y_j) + \mu(x_{j+1}, y_{j+1})$. (In other words, $L(x, y)$ is obtained by linear extension of the values L_j.) Then it is clear that

$$L^{\#}(\alpha, \beta) = \inf_{\xi, \eta \in \mathbb{R}} \{\sup_{(x, y) \in \Gamma_\Pi} \{\xi(x - \alpha) + \eta(y - \beta) + L(x, y)\}\}$$

and that $L^{\#}$ is a concave function such that $L^{\#} \geqslant L$ on the set Γ_Π. Moreover, $L^{\#}$ is the least function enjoying this property. Hence, if $(\alpha, \beta) = \sum \lambda_j (x_j, y_j)$, $\sum \lambda_j = 1$, $\lambda_j \geqslant 0$, then

$$L^{\#}(\alpha, \beta) \geqslant \sum \lambda_j L^{\#}(x_j, y_j) \geqslant \sum \lambda_j L(x_j, y_j)$$

and, thus,

$$L^{\#}(\alpha, \beta) \geqslant \sup \sum \lambda_j L(x_j, y_j).$$

The function in the right hand side is concave. We conclude that

$$L^{\#}(\alpha, \beta) = \sup \sum \lambda_j L(x_j, y_j) = \sup \sum \lambda_j L_j.$$

Here the supremum is taken over all decompositions $(\alpha, \beta) = \sum \lambda_j (x_j, y_j)$.

It remains to study the supremum

$$\sup\{\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n\} \tag{4.1}$$

under the constraints

$$\begin{aligned} \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n &= \alpha, \\ \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n &= \beta, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_n &= 1, \\ \lambda_j &\geqslant 0, \end{aligned} \tag{4.2}$$

However, it is well-known (theorem of Carathéodory) that maximum is attained in (4.1) at the vertices of the convex set in $\mathbb{R}^n$ defined by the points $(\lambda_1, \lambda_2, ..., \lambda_n)$ satisfying (4.2) and that these points have at most three components different from 0. This means that when the maximum is attained the system (4.2) reduces to a 3×3 system, say, with the solution $\lambda_i = c_i$, $\lambda_j = c_j$ and $\lambda_k = c_k$. This gives

$$\sup(\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_1 L_n) = \max_{i<j<k} (c_i L_i + c_j L_j + c_k L_k),$$

where $c_i, c_j, c_k \geqslant 0$. Hence

$$D_{\alpha\beta} = \max\{M_i^{c_1(i,j,k)}, M_j^{c_2(i,j,k)}, M_k^{c_3(i,j,k)}\},$$

completing the proof. ∎

4.3. *The Trigonometric Case* $S = S_{\mathbb{T}}$

Let the circle $\mathbb{T}$ be divided into two parts Γ_0 and Γ_1 by the line $l = \{y = b\}$ and let D_0 and D_1 be the corresponding parts of the unit disc (see Fig. 4.1).

We consider the case when $M(\gamma) = M_0$ on Γ_0 and $M(\gamma) = M_1$ on Γ_1 with $M_0 \leqslant M_1$.

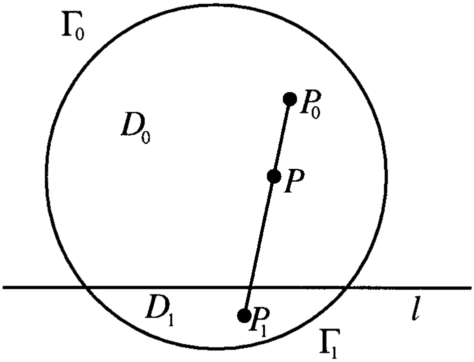

FIGURE 4.1

THEOREM 4.2. *Let $P = \{\alpha, \beta\}$ belong to the unit disc. Then the value of $D^{S_T}_{(\alpha,\beta)}(M)$ is given by*

$$\begin{cases} M_1, & \beta < b, \\ M_1^{1-\eta} M_0^{\eta},\ \eta = \dfrac{\beta - b}{1-b}, & \beta > b,\ \dfrac{1-\beta}{1-b} > \dfrac{|\alpha|}{a}, \\ M_1^{1-\mu} M_0^{\mu},\ \mu = \dfrac{(a-|\alpha|)^2 + (\beta-b)^2}{2\,[1 - a\,|\alpha| - b\beta]}, & \beta > b,\ \dfrac{1-\beta}{1-b} < \dfrac{|\alpha|}{a}. \end{cases}$$

For the definition of a see Remark 4.1 and Fig. 4.2.

Remark 4.1. The four different cases in Theorem 4.2 corresponds to the four different subsets of the unit disc (see Fig. 4.2).

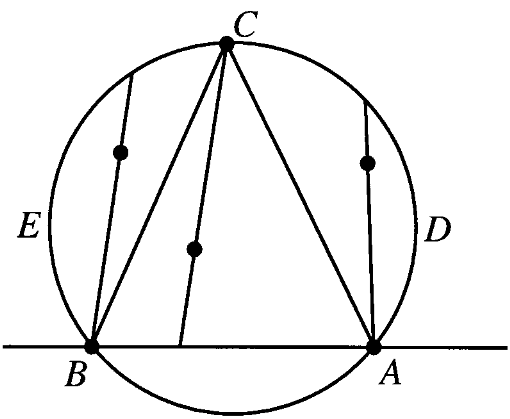

FIGURE 4.2

Here $A=(a, b)$ and $B=(-a, b)$, where $a>0$, are the two points where the line l intersects $\mathbb{T}$ and $C=(0, 1)$. The first case corresponds to the area D_1 below the line l, the second case to the area inside the triangle ABC, the third case to the segment ACD (the case $\alpha>0$) and the fourth to the segment BCE (the case $\alpha<0$).

Proof. Let $L^{\#}$ be defined as in the proof of Theorem 4.1. For the sake of simplicity we first consider the case when $M_0=1$ and $M_1=e$ (so that $L_0=0$ and $L_1=1$). This means that $L^{\#}$ is the smallest concave function equals 1 on Γ_1 and 0 on Γ_0. We note that as before it is sufficient to prove that

$$L^{\#}(P)=\begin{cases} 1, & \beta<b, \\ 1-\dfrac{\beta-b}{1-b}, & \beta>b, \dfrac{1-\beta}{1-b}>\dfrac{|\alpha|}{a}, \\ 1-\dfrac{(a-|\alpha|)^2+(\beta-b)^2}{2[1-a\,|\alpha|-b\beta]}, & \beta>b, \dfrac{1-\beta}{1-b}<\dfrac{|\alpha|}{a}. \end{cases} \tag{4.3}$$

Consider the line segment P_0P_1 containing the point $P=(\alpha, \beta)$ such that $P_0\in D_0$ and $P_1\in D_1$. Then we can write $P=(1-\theta)\,P_0+\theta P_1$, where $0\leqslant\theta\leqslant 1$. It is clear that if we choose P_0P_1 in such a way that θ is a maximum, then θ is the value of $L^{\#}(P)$. Moreover, when solving the maximum problem, we can assume that $P_0\in\Gamma_0$ and $P_1\in D\cap l$. We have to distinguish several cases.

Case 1. $P\in D_1$. Then we can take $P_1=P$ so that $\theta=1$ and $L^{\#}(P)=1$.

Case 0. $P\in D_0$. Let

$$\begin{cases} x=\alpha_1+\tau(\alpha-\alpha_1), \\ y=\beta_1+\tau(\beta-\beta_1), \end{cases}$$

be the equation of the straight line through $P_0=(\alpha_0, \beta_0)$ and $P_1=(\alpha_1, \beta_1)$. We note that P_0, P and P_1 correspond to the parametervalues $(\beta_0-b)/(\beta-b)$, 1 and 0, respectively. This means that $\theta=(\beta_0-\beta)/(\beta_0-b)$. From here it is seen that we must determine P_0 in such a way that β_0 becomes a maximum. Obviously, we have to divide up this case into further cases:

Case 0_1. P lies inside the triangle ABC. Then we can take $P_0=C$ so that $\beta_0=1$ and we conclude that $L^{\#}(P)=(1-\beta)/(1-b)$.

Case 0_2. P lies inside the segment ACD. Then we can take $P_1 = A$, that is, $\alpha_1 = a$. The equation of the line through P_0 and P_1 can also be written as

$$\begin{cases} x = a + \tau(\alpha - a), \\ y = b + \tau(\beta - b). \end{cases}$$

Moreover, we have to cut this line with $\mathbb{T}$: We find

$$1 = x^2 + y^2 = 1 + 2\tau[a(\alpha - a) + b(\beta - b)] + \tau^2[(\alpha - a)^2 + (\beta - b)^2]$$

yielding

$$\tau = \tau_0 = \frac{2(1 - a\alpha - b\beta)}{(\alpha - a)^2 + (\beta - b)^2} \qquad \text{and} \qquad \tau = \tau_1 = 0.$$

Therefore, as before, we find

$$\theta = 1 - \frac{1}{\tau_0} = L^{\#}(P) = \frac{1 - \alpha^2 - \beta^2}{2(1 - a\alpha - b\beta)}.$$

Case 0_3. P lies inside the segment BCE. This case follows from Case 0_2 by reflection in the y-axis which means that only the sign of α will be changed, i.e.,

$$Ł^{\#}(P) = \frac{1 - \alpha^2 - \beta^2}{2(1 + a\alpha - b\beta)}.$$

Summing up the results in the cases 1, 0_1, 0_2 and 0_3 we obtain (4.3). The proof of the general case $M_0 \leqslant M_1$ can step by step be carried out in the same way. The only difference is that we must all the time work with the correspondingly modified and somewhat longer expression for $L^{\#}(P)$. ∎

Remark 4.2. The method of proof of Theorem 4.2 can obviously be applied in similar situations in higher dimensions. For instance, the result of Theorem 4.2 extends with only verbal changes to the case of the ball in $\mathbb{R}^3$.

5. EXAMPLES

5.1. *The Case when A is a Banach couple* (A_0, A_1)

Let us take $A(\gamma) = A_0$ on $\Gamma_0 = [-(\pi/2), (\pi/2))$ and $A(\gamma) = A_1$ on $\Gamma_1 = [(\pi/2), (3\pi/2))$.

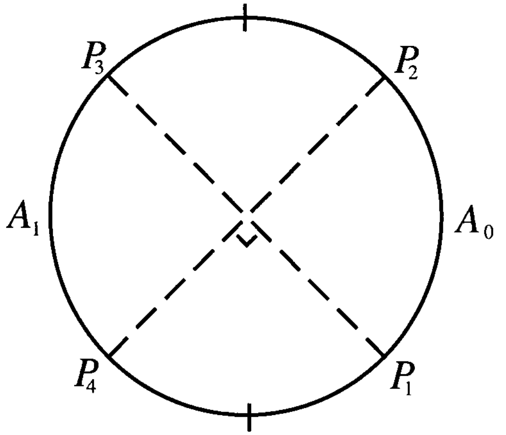

FIGURE 5.1

EXAMPLE 5.1. We set $\Pi_1 = \{(0, -1), (0, 1)\}$, $\Pi_2 = \{(0, -1), (1, 0), (-1, 0), (0, 1)\}, ...$; quite generally, Π_N is obtained by adding to Π_{N-1} the midpoints of all arcs at step $N-1$. In the Sparr limit case we obtain

$$(A)^{S^1_{\mathrm{d}}}_{0, p; J} = (A)^{S^1_{\mathrm{d}}}_{0, p; K} = (A_0, A_1)_{\theta, p} \qquad \text{with} \quad \theta = |\Gamma_1|_0 = \tfrac{1}{2}.$$

Moreover, as $S^1_{\mathrm{d}} \subset S^N_{\mathrm{d}}$ for every $N > 1$, we have

$$(A_0, A_1)_{(1/2), p} \subset (A)^{S^N_{\mathrm{d}}}_{0, p; J} \subset (A)^{S^N}_{0, p; K} \subset (A_0, A_1)_{(1/2), p}.$$

Therefore, by Theorem 3.2,

$$(A)^{S_{\mathrm{d}}}_{0, p; K} = (A_0, A_1)_{(1/2), p} \qquad \text{for all} \quad p \geqslant 1.$$

EXAMPLE 5.2. Now we choose a different family of collections of points $\{\Pi_N\}_N$ obtained as follows:

We let $\Pi_1 = \{P_1, P_2, P_3, P_4\}$ such that $(0, 1)$ is the midpoint of the arc $\widehat{P_2 P_3}$ and $(0, -1)$ is the midpoint of the arc $\widehat{P_4 P_1}$ (see Fig. 5.1).

Moreover, we assume that $A_0 \subset A_1$ and write

$$\theta = \bar{\theta} = (\theta_2, \theta_3, \theta_4) = (|\widehat{P_2 P_3}|_0, |\widehat{P_3 P_4}|_0, |\widehat{P_4 P_1}|_0).$$

Then, according to Theorem 3.1,

$$\begin{aligned}(A)^{S^1_{\mathrm{d}}}_{0, p; K} &= (A_0, A_0 + A_1, A_1, A_0 + A_1)_{\theta, p; K} \\ &= (A_0, A_1, A_1, A_1)_{\theta, p; K} \subset (A_0, A_1)_{\theta_1, p},\end{aligned}$$

where $\theta_1 = \theta_2 + \theta_3 + \theta_4$. On the other hand, we have

$$(A_0, A_1)_{\theta_1, p} \subset (A_0, A_1, A_1, A_1)_{\theta, p; J},$$

and, hence,

$$(A)^{S_d^1}_{0, p; K} = (A_0, A_1)_{\theta_1, p}.$$

Next, Π_N is defined recursively by adding a new point in every arc in the $(N-1)$st step with the only restriction that the points $(0, 1)$ and $(0, -1)$ do not belong to any Π_N. Arguing as above we find

$$(A)^{S_d^N}_{0, p; K} = (A_0, A_1)_{\theta_N, p} \qquad \text{where} \quad \theta_N > \theta_{N-1} \quad \text{and} \quad \theta_N \to \tfrac{1}{2} \quad \text{as} \quad N \to \infty.$$

Therefore in view of Theorem 3.2 we obtain

$$(A)^{S_d}_{0, p; K} = \bigcap_N (A_0, A_1)_{\theta_N, p}.$$

Remark 5.1. Comparing the results in Examples 5.1 and 5.2 we see that $(A)^{S_d}$ depends in an essential way on the choice of the family $\{\Pi_N\}_N$, because in general

$$(A_0, A_1)_{(1/2), p} \neq \bigcap_N (A_0, A_1)_{\theta_N, p}$$

no matter how the sequence $\{\theta_N\}$ with $\theta_N < \frac{1}{2}$, $\theta_N \to \frac{1}{2}$ is chosen. (A counter-example is obtained by choosing $p = 2$, $A_0 = l_1$, $A_1 = l_\infty$.)

EXAMPLE 5.3. For the set S_T it is known that (see [3])

$$K_1(\alpha_{n,m}, a) = \inf_{a = a_0 + a_1} \{\inf_{\Gamma_0} \alpha_{n,m}(\gamma) \, \|a_0\|_{A_0} + \inf_{\Gamma_1} \alpha_{n,m}(\gamma) \, \|a_1\|_{A_1}\}.$$

Thus, in each case we have an explicit description of the space $(A)^{S_T}_K$. For example for the couple $(A_0, A_1) = (L_1, L_\infty)$ we can use the usual formula $K(t, f) = \int_0^t f^*(u) \, du$ to obtain that

$$(A)^{S_T}_{0, p; K} = \left\{ f \in L^1 + L^\infty : c^1_{n,m} f^{**}\left(\frac{c^1_{n,m}}{c^0_{n,m}}\right) \in l^p \right\},$$

where $f^{**}(t) = (1/t) \int_0^t f^*(u) \, du$ and $c^0_{n,m} = \inf_{\Gamma_0} \alpha_{n,m}$ and $c^1_{n,m} = \inf_{\Gamma_1} \alpha_{n,m}$.

5.2. *Interpolation of Families of Weighted Spaces*

The following example is fundamental in [5] where an extension of it is also proved.

EXAMPLE 5.4. Let $(F_\alpha)_{\alpha \in S}$ be a collection of Banach spaces and let F_α^γ be the space F_α endowed with the norm $\|\cdot\|_{F_\alpha^\gamma} = \alpha^{-1}(\gamma)\ \|\cdot\|_{F_\alpha}$ and set

$$l_p(F_\alpha^\gamma) = \left\{ (x_\alpha)_{\alpha \in S};\ \left(\sum_{\alpha \in S} \|x_\alpha\|_{F_\alpha^\gamma}^p \right)^{1/p} < \infty \right\}.$$

Then, for all $1 \leqslant p(\gamma),\ p \leqslant \infty$,

$$(l_{p(\gamma)}(F_\alpha^\gamma))^S_{z_0,\, p;\, K} = (l_{p(\gamma)}(F_\alpha^\gamma))^S_{z_0,\, p;\, J} = l_p(F_\alpha^{\alpha(z_0)}),$$

where $F_\alpha^{\alpha(z_0)}$ is the space F_α endowed with the norm $\alpha^{-1}(z_0)\|\cdot\|_{F_\alpha}$.

Proof. It is enough to show that

$$l_p(F_\alpha^{\alpha(z_0)}) \subset (l_{p(\gamma)}(F_\alpha^\gamma))^S_{z_0,\, p;\, J} \subset (l_{p(\gamma)}(F_\alpha^\gamma))^S_{z_0,\, p;\, K} \subset l_p(F_\alpha^{\alpha(z_0)}),$$

where the middle embedding follows from the general theory (see [3, 4]).

In order to establish the first embedding, let $b = (b_\alpha)_{\alpha \in S} \in l_p(F_\alpha^{\alpha(z_0)})$ and write $u_\alpha = b_\alpha \delta_\alpha$ where $\delta_\alpha = (0, ..., 0, \overset{(\alpha)}{1}, 0, ...)$. Then $b = \sum_\alpha u_\alpha$ with $u_\alpha \in \bigcap_{\gamma \in \mathbb{T}} l_1(F_\alpha^\gamma)$ and

$$J(\alpha, u_\alpha) = \int_{\mathbb{T}} \alpha(\gamma)\ \|u_\alpha\|_{F_\alpha^\gamma}\, d\gamma = \|b_\alpha\|_{F_\alpha},$$

and, therefore,

$$\|b\|_{(l_1(F_\alpha^\gamma))^S_{z_0,\, p;\, J}} \leqslant \left(\sum_\alpha \left(\frac{J(\alpha, u_\alpha)}{\alpha(z_0)} \right)^p \right)^{1/p}$$

$$= \left(\sum_\alpha (\alpha^{-1}(z_0)\ \|b_\alpha\|_{F_\alpha})^p \right)^{1/p} = \|b\|_{l_p(F_\alpha^{\alpha(z_0)})}.$$

For the last embedding, let $b = (b_\alpha)_{\alpha \in S} \in (l_\infty(F_\alpha^\gamma))^S_{z_0,\, p;\, K}$. Then, there exists $b(\gamma) = (b_\alpha(\gamma))_{\alpha \in S}$ such that $b = \int_{\mathbb{T}} b(\gamma)\, d\gamma$ and, hence,

$$\|b_\alpha\|_{F_\alpha} \leqslant \int_{\mathbb{T}} \|b_\alpha(\gamma)\|_{F_\alpha}\, d\gamma = \int_{\mathbb{T}} \alpha(\gamma)\ \|b_\alpha(\gamma)\|_{F_\alpha^\gamma}\, d\gamma$$

$$\leqslant \int_{\mathbb{T}} \alpha(\gamma)\ \|(b_\alpha(\gamma))_\alpha\|_{l_\infty(F_\alpha^\gamma)}\, d\gamma.$$

Taking the infimum over all possible representations of b_α we obtain

$$\|b_\alpha\|_{F_\alpha} \leqslant K(\alpha, b; \{l_\infty(F_\alpha^\gamma)\}) = K(\alpha, b)$$

and, thus,

$$\|b\|_{l_p(F_\alpha^{\alpha(z_0)})} = \left(\sum_\alpha (\alpha^{-1}(z_0)\ \|b_\alpha\|_{F_\alpha})^p\right)^{1/p} \leqslant \left(\sum_\alpha \left(\frac{K(\alpha, b)}{\alpha(z_0)}\right)^p\right)^{1/p}$$
$$= \|b\|_{(l_\infty(F_\alpha^\gamma))^S_{z_0, p; K}}. \quad \blacksquare$$

Taking $F_\alpha = \mathbb{C}$ for every $\alpha \in S$ one obtains the following result: *For all* $1 \leqslant p$, $p(\gamma) \leqslant \infty$ *and* $S = \{\alpha\}_\alpha$, *we have*

$$(l_{p(\gamma)}(\alpha^{-1}(\gamma)))^S_{z_0, p; K} = (l_{p(\gamma)}(\alpha^{-1}(\gamma)))^S_{z_0, p; J} = l_p(\alpha^{-1}(z_0)).$$

Remark 5.2. In particular, the statement above implies that

$$l_{p(\gamma)}(2^{-n\cos\gamma - m\sin\gamma}))^{S_\mathbb{T}}_{z_0, p; J} = l_{p(\gamma)}(2^{-n\cos\gamma - m\sin\gamma)}))^{S_\mathbb{T}}_{z_0, p; K}$$
$$= l_p(2^{-n\alpha - m\beta}).$$

Moreover, if $p(\gamma) = 1$ Cwikel and Janson [11] have proved that $U_M(A, Z)$, where M is the Sparr K-method with parameter p, equals $l_1(2^{-n\alpha - m\beta})$. Hence, Theorem 3.5 implies that

$$l_1(2^{-n\alpha - m\beta}) = U_M(A, Z) \subset (A)^{S_\mathrm{d}}_{z_0, 1; K} \subset (A)^{\hat{S}_\mathrm{p}}_{z_0, 1; K} \subset (A)^{S_\mathbb{T}}_{z_0, 1; K}$$
$$= l_1(2^{-n\alpha - m\beta}).$$

Furthermore, according to Remark 3.5 (cf. Theorem 3.6) we also have

$$l_1(2^{-n\alpha - m\beta}) \subset (A)^{S_\mathbb{T}}_{z_0, 1; J} \subset (A)^{\hat{S}_\mathrm{p}}_{z_0, 1; J} \subset (A)^{S_\mathrm{d}}_{z_0, 1; J} \subset A[z_0] = l_1(2^{-n\alpha - m\beta}).$$

We conclude that all the spaces $U_M(A, Z)$, $(A)^{S_\mathrm{d}}_{z_0, 1; K}$, $(A)^{\hat{S}_\mathrm{p}}_{z_0, 1; K}$, $(A)^{S_\mathbb{T}}_{z_0, 1; K}$, $(A)^{S_\mathbb{T}}_{z_0, 1; J}$, $(A)^{\hat{S}_\mathrm{p}}_{z_0, 1; J}$, $(A)^{S_\mathrm{d}}_{z_0, 1; J}$, $A[z_0]$ and $l_1(2^{-n\alpha - m\beta})$ coincide.

EXAMPLE 5.5. Consider $A(\gamma) = L^\infty(W(\gamma, \cdot))$, where

$$0 < u(x) \leqslant \inf_\gamma W(\gamma, x) \leqslant \sup_\gamma W(\gamma, x) \leqslant v(x) < \infty.$$

Then $A(\gamma)$ is a bounded family when $\bigcap_{\gamma \in \mathbb{T}} A(\gamma) \neq 0$ and, moreover, we have

$$(L^\infty(W(\gamma, \cdot)))^S_{z_0, \infty; K} \subset L^\infty(D_S^{-1}(W^{-1}(\gamma, \cdot))), \tag{5.1}$$

where

$$D_S^{-1}(W^{-1}(\gamma,\cdot)) = \sup_{\alpha} \inf_{\gamma\in\mathbb{T}} \frac{\alpha(z_0)\, W(\gamma,x)}{\alpha(\gamma)}.$$

Proof of (5.1). We note that by Theorem 2.4 it suffices to prove (5.1) in the case of the K_1-functional. Let $f\in(L^\infty(W(\gamma,\cdot)))^{S,1}_{z_0,\infty;K}$. Writing $f(x)=\sum_\gamma f(\gamma,x)$ we have for every $\alpha\in S$

$$|f(x)| \inf_{\gamma\in\mathbb{T}} \frac{W(\gamma,x)}{\alpha(\gamma)} \leqslant \sum_\gamma \frac{W(\gamma,x)|f(\gamma,x)|}{\alpha(\gamma)} \leqslant \sum_\gamma \frac{\|f(\gamma,\cdot);L^\infty(W(\gamma,\cdot))\|}{\alpha(\gamma)},$$

whence by the definition of the K_1-functional

$$|f(x)| \inf_{\gamma\in\mathbb{T}} \frac{W(\gamma,x)}{\alpha(\gamma)} \leqslant K(\alpha^{-1},f).$$

It follows that

$$\|f;L^\infty(D_S^{-1}(W^{-1}(\gamma,\cdot)))\| = \sup_x \left(|f(x)| \sup_\alpha \inf_{\gamma\in\mathbb{T}} \frac{\alpha(z_0)\,W(\gamma,x)}{\alpha(\gamma)}\right)$$

$$\leqslant \sup_\alpha \frac{K(\alpha^{-1},f)}{\alpha^{-1}(z_0)} \leqslant \|f;(L^\infty(W(\gamma,\cdot)))^S_{z_0,\infty;K}\|. \quad \blacksquare$$

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